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Hausdorff dimension of unions of affine subspaces and related questions

Theses of the dissertation

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1 Introduction

Investigating unions of lines or unions of higher dimensional affine subspaces has been an important field of geometric measure theory in the past decades. Studying them, one can reveal connections between largeness and structure of these geometric objects, which is a very natural goal for mathematicians.

The most significant example for such investigations is the study of Besicovitch sets. We say that $B \subset \mathbb{R}^n$ is a Besicovitch set if for each direction $e \in S^{n-1}$, there exists a unit line segment pointing to that direction contained in B. In 1928, A. S. Besicovitch [1] proved that there exist Besicovitch sets of Lebesgue measure zero. The well known Kakeya conjecture formulates that the Hausdorff dimension of Besicovitch sets in $\mathbb{R}^n$ must be n. The conjecture was solved in the plane by R. O. Davies in 1971, [3], but it is open in higher dimensions ($n \geq 3$). Several attempts have been made to estimate the dimension of Besicovitch sets from below, the best known Hausdorff dimension bounds which have been obtained using very powerful methods (see [20], [10], and [11]), have order between $0.5 \cdot n$ and $0.6 \cdot n$ which are still far from the conjectured full dimension n. For a survey of results related to the Kakeya conjecture, see [14].

Although the Kakeya problem seems to be a question of purely geometric nature, many fundamental problems in harmonic analysis, arithmetic geometry, and number theory are closely related to it, see e.g. [18] or [14]. The various attempts towards solving the Kakeya conjecture have enriched the world of mathematics considerably in the last almost 50 years. This gives hope that investigating related problems can also reveal important connections, and thus are worth to study.

In the dissertation we consider general unions of lines and unions of affine subspaces, without any prescribed geometric condition on the position of the subspaces. The space of k-dimensional affine subspaces in $\mathbb{R}^n$ admits a natural metric, thus we can consider the Hausdorff dimension of a family of affine subspaces. We are interested in the following question: how large is the Hausdorff dimension of a union of k-dimensional affine subspaces in $\mathbb{R}^n$ as a function of k, n, and the Hausdorff dimension of the family from which the subspaces are taken? Such results were first proved by D. M. Oberlin in 2014, see [15]. We present several new results towards answering this question. Our results are based on and supplemented by the study of some closely related other problems. The dissertation is based on [H2], on the papers [H3] and [H4] which are joint work with Tamás Keleti and András Máthé, and on [H1], which is a joint work with Alan Chang, Marianna Csörnyei, and Tamás Keleti.

2 Unions of affine subspaces

In Chapter 2 of the thesis we give a summary of our results about the Hausdorff dimension of unions of affine subspaces depending on the Hausdorff dimension of the family E from which the subspaces are taken. Most of our results for unions of affine subspaces follow from more general theorems, namely, from estimates for the Hausdorff dimension of Furstenberg-type sets associated to families of affine subspaces, which are extensively studied in Chapters 3, 4, and 5 of the thesis.

Fix $1 \le k < n$ integers, let $G(n,k)$, and $A(n,k)$ denote the space of all k-dimensional linear, and affine subspaces of $\mathbb{R}^n$, respectively. For $P_i = V_i + a_i \in A(n,k)$, where $V_i \in G(n,k)$ and $a_i \in V_i^{\perp}$, $i = 1,2$, we put $d(P_1, P_2) = \|\pi_{V_1} - \pi_{V_2}\| + |a_1 - a_2|$, where $\pi_{V_i} : \mathbb{R}^n \to V_i$ denotes the orthogonal projection onto V_i ($i = 1,2$), and $\|\cdot\|$ denotes the standard operator norm. Then d is a metric on $A(n,k)$, see [13].

Definition. Let ρ be a metric on $A(n,k)$. We say that ρ is a *natural metric*, if ρ and the above d are strongly equivalent, that is, if there exist positive constants K_1 and K_2 such that, for every $P_1, P_2 \in A(n,k)$, $K_1 \cdot d(P_1,P_2) \le \rho(P_1,P_2) \le K_2 \cdot d(P_1,P_2)$.

We investigate unions $B = \bigcup_{P \in E} P \subset \mathbb{R}^n$ of affine subspaces, where $\emptyset \ne E \subset A(n,k)$. Fixing a natural metric ρ on $A(n,k)$, we can consider the Hausdorff dimension of E in the metric space $(A(n,k), \rho)$. Here and in the sequel the notation $\dim$ will always refer to Hausdorff dimension. Our main question is the following.

Question 2.1. *Let $\emptyset \ne E \subset A(n,k)$, and $B = \bigcup_{P \in E} P$. What can we say about $\dim B$ as a function of k, n, and $\dim E$?*

As it often happens when investigating Hausdorff dimension of sets, it is easier to give an upper bound for $\dim B$ than to give a lower bound. The following lemma contains a very natural upper bound for $\dim B$.

Lemma 2.2.1. *Let $\emptyset \ne E \subset A(n,k)$, and $B = \bigcup_{P \in E} P$. Then*

$$\dim B \le \dim E + k.$$

Our first lower bound for the Hausdorff dimension of unions of affine subspaces, which is a joint result with Tamás Keleti and András Máthé, see [H3], is the following.

Theorem 2.2.5. *Let $\emptyset \ne E \subset A(n,k)$, and put $B = \bigcup_{P \in E} P \subset \mathbb{R}^n$. Then*

$$\dim B \ge k + \min\{\dim E, 1\}.$$

We remark that for $k = n-1$, a result of Oberlin [15], as well as a result of Falconer and Mattila [5] implies Theorem 2.2.5, but for $k < n-1$, it is new.

Combining Lemma 2.2.1 and Theorem 2.2.5 we immediately obtain the following.

Theorem 2.2.6. *Let $\emptyset \neq E \subset A(n,k)$ with $\dim E \in [0,1]$, and put $B = \bigcup_{P \in E} P \subset \mathbb{R}^n$. Then*

$$\dim B = k + \dim E. \tag{2.1}$$

We also prove the following second lower bound about unions of affine subspaces, based on [H2].

Theorem 2.2.8. *Let $\emptyset \neq E \subset A(n,k)$, and put $B = \bigcup_{P \in E} P \subset \mathbb{R}^n$. Then*

$$\dim B \geq k + \frac{\dim E}{k+1}.$$

We remark that Theorem 2.2.5 is sharp if $\dim E \leq k+1$, Theorem 2.2.8 is stronger than Theorem 2.2.5 if and only if $\dim E > k+1$, and Theorem 2.2.8 is sharp if $\dim E = m(k+1)$, where $m \in [0, n-k]$ is any integer.

3 Furstenberg-type sets associated to families of affine subspaces

In Chapter 3 of the thesis we introduce the notion of Furstenberg-type sets associated to families of affine subspaces. The classical Furstenberg set problem, which originates from the work of H. Furstenberg [7], is the following. Fix $0 < \alpha \leq 1$, and suppose that $F \subset \mathbb{R}^2$ is a compact set such that for each direction $e \in S^1$ there is a line L_e pointing to that direction such that $\dim (L_e \cap F) \geq \alpha$, where $\dim$ denotes the Hausdorff dimension. What is the smallest possible value of $\dim F$ as a function of α? Such sets are called α-Furstenberg-sets. In 1999, T. Wolff gave the following partial answers to the question, see [21]. For any $0 < \alpha \leq 1$, if $F \subset \mathbb{R}^2$ is an α-Furstenberg set, then $\dim F \geq 2\alpha$, and $\dim F \geq \alpha + 1/2$. Moreover, for any $0 < \alpha \leq 1$ there exists an α-Furstenberg set with $\dim F = 3\alpha/2 + 1/2$. In the $\alpha = 1/2$ case J. Bourgain [2] improved the lower bound 1 to $\dim F \geq 1 + c$ for some absolute constant $c > 0$ using the work of N. Katz and T. Tao [9]. However, the smallest possible value of the Hausdorff dimension of Furstenberg sets is still unknown.

We introduce the following generalized definition. Let $1 \leq k < n$ integers, and recall that $A(n,k)$ denotes the space of all k-dimensional affine subspaces of $\mathbb{R}^n$. Let $0 < \alpha \leq k$, and $0 \leq s \leq (k+1)(n-k)$ be any real numbers.

Definition. We say that $B \subset \mathbb{R}^n$ is an (α, k, s)-Furstenberg set, if there exists $\emptyset \neq E \subset A(n,k)$ with $\dim E = s$ such that $\dim(B \cap P) \geq \alpha$ for all $P \in E$.

Our first estimate, which is a joint result with Tamás Keleti and András Máthé, see [H3], is the following.

Theorem 3.2.1. *Let $0 < \alpha \leq k$, and $0 \leq s \leq (k+1)(n-k)$. If $B \subset \mathbb{R}^n$ is an (α, k, s)-Furstenberg set, then*

$$\dim B \geq 2\alpha - k + \min\{s, 1\}.$$

This estimate is nontrivial only if $\alpha > k-1$. Our second estimate, which also captures a more interesting behaviour in the $\alpha > k-1$ case, is the following, see [H2]:

Theorem 3.3.4. *Let $0 < \alpha \leq k$, and $0 \leq s \leq (k+1)(n-k)$. If $B \subset \mathbb{R}^n$ is an (α, k, s)-Furstenberg-set, then*

$$\dim B \geq \alpha + \frac{s - (k - \lceil \alpha \rceil)(n-k)}{\lceil \alpha \rceil + 1},$$

where $\lceil \alpha \rceil$ denotes the least integer greater than or equal to α.

As we mentioned before, our results for Furstenberg-type sets associated to families of affine subspaces naturally imply some of our results about unions of affine subspaces, namely, Theorem 3.2.1 implies Theorem 2.2.5, and Theorem 3.3.4 implies Theorem 2.2.8. The proofs of Theorems 3.2.1 and 3.3.4 can be found in Chapters 3, 4, and 5 of the thesis.

4 A Fubini-type theorem for Hausdorff dimension

In Chapter 6 of the thesis we prove Fubini-type results for Hausdorff dimension based on a joint work with Tamás Keleti and András Máthé, see [H4]. It is well known that for Hausdorff dimension, the classical Fubini theorem does not hold. More precisely, it is not true in general that if all the vertical sections of a plane set are β-dimensional then the set is $(\beta+1)$-dimensional, its Hausdorff dimension can be strictly bigger than $\beta+1$. We prove a modified Fubini-type theorem. As before, dim denotes Hausdorff dimension, and for any $s \geq 0$, let $\mathcal{H}^s$ denote the s-dimensional Hausdorff measure.

Fix two integers $n, k \geq 1$, let $B \subset \mathbb{R}^k \times \mathbb{R}^n$, and for any $t \in \mathbb{R}^k$ let

$$B_t = \{x \in \mathbb{R}^n : (t, x) \in B\}.$$

For any $B \subset \mathbb{R}^k \times \mathbb{R}^n$, let $A = \operatorname{proj}_{\mathbb{R}^k} B \subset \mathbb{R}^k$, $\alpha = \dim A$, and assume that $\alpha > 0$. If $\mathcal{H}^\alpha(A) > 0$, then we define

$$\operatorname{ess}^\alpha\text{-}\sup(\dim B_t) = \sup\{q \geq 0 : \mathcal{H}^\alpha(\{t \in \mathbb{R}^k : \dim B_t > q\}) > 0\}.$$

If $\mathcal{H}^\alpha(A) = 0$, then we let

$$\operatorname{ess}^\alpha\text{-}\sup(\dim B_t) = \lim_{\varepsilon \to 0+} \operatorname{ess}^{\alpha\text{-}\varepsilon}\text{-}\sup(\dim B_t).$$

A classical theorem about Hausdorff measures (see e.g. [4]) implies that if B is as above, then $\dim B \geq \alpha + \operatorname{ess}^\alpha\text{-}\sup(\dim B_t)$. This means that if a set B has many large sections then B is large. We introduce the notion of Γ-null sets.

Definition (Γ_α-null sets)**.** For any $0 < \alpha \leq k$, we say that $G \subset \mathbb{R}^k \times \mathbb{R}^n$ is Γ_α-*null*, if for any $f : \mathbb{R}^k \to \mathbb{R}^n$ Lipschitz map,

$$\mathcal{H}^\alpha(t \in \mathbb{R}^k : (t, f(t)) \in G) = 0.$$

Let us recall the definition of Ahlfors regular sets (see e.g. [13]).

Definition (Ahlfors-regular sets)**.** For any $0 < \alpha \leq k$, we say that the Borel set $T \subset \mathbb{R}^k$ is *α-dimensional Ahlfors-regular*, if there exist $c, c' > 0$ such that for all $t \in T$, $0 < r \leq 1$,

$$c \cdot r^\alpha \leq \mathcal{H}^\alpha(T \cap B(t, r)) \leq c' \cdot r^\alpha.$$

By a σ-compact set we mean a set which can be written as a countable union of compact sets. We prove the following Fubini-type theorem about Hausdorff dimension.

Theorem 6.1.9. *Let $0 < \alpha \leq k$, $B \subset T \times \mathbb{R}^n$ be non-empty σ-compact, where $T \subset \mathbb{R}^k$ is an α-dimensional Ahlfors-regular Borel set, and assume that $\mathcal{H}^\alpha(\operatorname{proj}_{\mathbb{R}^k} B) > 0$. Then there exists a Borel set $G \subset B$ such that G is Γ_α-null, and* $\dim(B \setminus G) = \alpha + \operatorname{ess}^\alpha\text{-}\sup(\dim B_t)$.

The proof of Theorem 6.1.9 is based on the above mentioned classical result formulating that the existence of many large sections implies largeness, as well as the following theorem.

Theorem 6.1.11. *Let $0 < \alpha \leq k$, $B \subset T \times \mathbb{R}^n$ be non-empty σ-compact, where $T \subset \mathbb{R}^k$ is an α-dimensional Ahlfors-regular Borel set. Assume that β is a real number such that $\dim B_t \leq \beta$ for all $t \in T$. Then there exists a Borel set $G \subset B$ such that G is Γ_α-null, and $\dim (B \setminus G) \leq \alpha + \beta$.*

We remark that the Ahlfors regularity condition in Theorems 6.1.9 and 6.1.11 is crucial, as it is proved in Chapter 6 of the thesis.

5 Applications of the results from Chapters 2, 3 and 6 of the thesis

Chapter 7 of the dissertation contains several applications for the results from Chapters 2, 3 and 6 of the thesis. The first application, which is a joint result with Tamás Keleti and András Máthé, see [H3], is concerned with the largeness of certain unions of skeletons of rotated unit cubes. As before, dim will denote Hausdorff dimension in this section.

Let $0 \leq k < n$ be integers. For a given cube $Q \subset \mathbb{R}^n$, by the k-skeleton of Q we mean the union of the k-dimensional faces of Q. Among other results, we prove the following.

Theorem 7.1.4. *Let $0 \leq k < n$ be integers, $0 \leq \alpha \leq k$ and $0 \leq r$ be real numbers, $\emptyset \neq C \subset \mathbb{R}^n$, and $B \subset \mathbb{R}^n$ be such that for every $x \in C$ there exists a k-dimensional affine subspace P at distance r from x such that P intersects B in a nonempty set of Hausdorff dimension at least α. Then $\dim B \geq 2\alpha - k + \dim C - (n-1)$.*

Specially, if B contains a k-dimensional affine subspace at a fixed positive distance from every point of $\mathbb{R}^n$, or if B contains the k-skeleton of a rotated unit cube centered at every point of $\mathbb{R}^n$, then $\dim B \geq k+1$.

As an application of our results from Chapters 3 and 6 of the thesis we derive Fubini-type results for the Hausdorff dimension of unions of affine subspaces as well as related projection theorems. These are based on a joint work with Tamás Keleti and András Máthé, see [H3] and [H4].

Let $2 \leq m, 1 \leq k < m$ be integers. Recall that $A(m,k)$ denotes the space of all k-dimensional affine subspaces of $\mathbb{R}^m$. We say that $P \in A(m,k)$ is *non-vertical*, if the projection of P onto $\mathbb{R}^k \times \{0\}$ is $\mathbb{R}^k \times \{0\}$. As before, by a σ-compact set we mean a set which can be written as a countable union of compact sets.

Corollary 7.2.6. *Fix a non-empty collection $E \subset A(m,k)$ of non-vertical k-dimensional affine subspaces such that $\dim E \le 1$, and $B = \bigcup_{P \in E} P \subset \mathbb{R}^m$ is σ-compact. Then $\dim B = \beta + k$, where $\beta = \text{ess}^k\text{-}\sup(\dim B_t)$, defined in the previous section.*

In particular, if $B = \bigcup_{P \in E} P$ is σ-compact with $\dim B < k+1$, then the above holds for B.

In the dissertation we formulate the conjecture that the above theorem holds for arbitrary unions of non-vertical k-dimensional affine subspaces. Our conjecture in the $k = 1$ case is closely related to the Kakeya conjecture.

We also prove two theorems for a restricted family of projections. Several authors investigated restricted families of projections in the last few years for different families of projections, see e.g. [8], [6], [17], [16]. Our proofs are based on our Fubini-type results as well as the standard duality connection between sections of unions of affine subspaces, and certain projections of the ,,code set" associated to the union.

Fix two integers $n, k \ge 1$, and fix $t = (t_1, \dots, t_k) \in \mathbb{R}^k$. Let

$$\pi_t : \mathbb{R}^{(k+1)n} \to \mathbb{R}^n, (x_0, x_1, \dots, x_k) \mapsto x_0 + t_1 \cdot x_1 + \dots + t_k \cdot x_k.$$

Let $\mathcal{L}^k$ denote the Lebesgue measure on $\mathbb{R}^k$.

Theorem 7.3.1. *Let $X \subset \mathbb{R}^{(k+1)n}$ be non-empty σ-compact with $\dim X \le 1$. Then for $\mathcal{L}^k$-almost all $t \in \mathbb{R}^k$,*

$$\dim \pi_t(X) = \dim X.$$

We also show the following further projection theorem.

Theorem 7.3.2. *Let $X \subset \mathbb{R}^{(k+1)n}$ be non-empty σ-compact. Then for $\mathcal{L}^k$-almost all $t \in \mathbb{R}^k$,*

$$\dim \pi_t(X) \ge \frac{\dim X}{k+1}.$$

We remark that Theorem 7.3.2 is sharp if $\dim X = (k+1)n$.

6 Unions of skeletons of polytopes

Chapter 8 of the thesis is devoted to the study of sets containing skeletons of polytopes based on a joint work with Alan Chang, Marianna Csörnyei, and Tamás Keleti, see [H1]. The study of such problems started in the work of T. Keleti, D. T. Nagy and P. Shmerkin, [12], and was continued by R.Thornton, [19]. Here we show that the smallest possible Hausdorff dimension of a set A which contains the k-skeleton of a rotated unit cube centered at every point of $\mathbb{R}^n$ is $k+1$. Surprisingly, we get the same result if we require skeletons of cubes of all scales centered at every point of $\mathbb{R}^n$. As we mentioned in the previous section (see Theorem 7.1.4), the fact that $k+1$ is a lower bound for the Hausdorff dimension of sets containing the k-skeleton of a rotated unit cube centered at every point of $\mathbb{R}^n$ follows from joint results with Tamás Keleti and András Máthé, see [H3]. The proof for sharpness is based on typicality arguments, formulated using an appropriate Baire space. As before, dim will denote Hausdorff dimension in this section.

Definition. By a rotated copy of a fixed set $S \subset \mathbb{R}^n$ we mean a set of the form $x + T(S) = \{x + T(s) \ : \ s \in S\}$, where $x \in \mathbb{R}^n$ and $T \in SO(n)$. We say that $x + T(S)$ is a rotated copy of S centered at x.

Similarly, for any $r \geq 0$ and $x \in \mathbb{R}^n$, by a rotated and scaled copy of S of scale r centered at x we mean a set of the form $x + r \cdot T(S) = \{x + r \cdot T(s) \ : \ s \in S\}$, where $T \in SO(n)$.

We prove the following theorem.

Theorem 8.2.1. *Let $0 \leq k < n$ be integers, and $S \subset \mathbb{R}^n$ with $\dim S = k$ that can be covered by a countable union of k-dimensional affine subspaces that do not contain 0. Then the minimal Hausdorff dimension of a Borel set $A \subset \mathbb{R}^n$ that contains*

(a) a rotated copy of S centered at every point of $\mathbb{R}^n$

(b) a rotated and scaled copy of S of every scale centered at every point of $\mathbb{R}^n$

is $k + 1$.

This easily implies that the minimal Hausdorff dimension of a set A containing the k-skeleton of a rotated and scaled cube of every positive size centered at every point of $\mathbb{R}^n$ is $k + 1$.

Investigating the $k = n - 1$ case, one can check that Theorem 7.1.4 implies the following. Let $S \subset \mathbb{R}^n$ be a set which can be covered by a countable union of $(n-1)$-dimensional affine subspaces such that $\dim S = n - 1$. If A contains a rotated copy of S centered at each point of $\mathbb{R}^n$, then A has Hausdorff dimension n. The next theorem states that such an A can have zero Lebesgue measure provided that $0 \notin S$, even if it contains rotated and scales copies of all scales centered at every point.

Theorem 8.2.3. *Let $S \subset \mathbb{R}^n$ $(n \geq 2)$ be a set that can be covered by countably many hyperplanes and suppose that $0 \notin S$. Then there exists a set of Lebesgue measure zero that contains a scaled and rotated copy of S of every scale centered at every point of $\mathbb{R}^n$.*

Theorem 8.2.3 immediately implies the following two statements. There exists a set of Lebesgue measure zero that contains a scaled and rotated copy of the boundary of a cube of every scale centered at every point of $\mathbb{R}^n$. There exists a set of Lebesgue measure zero in $\mathbb{R}^n$ which contains a hyperplane at every positive distance from every point as well as a punctured hyperplane through every point. The proof of Theorem 8.2.3 involves typicality arguments in the sense of Baire category, similarly to our result about Hausdorff dimension. Based on our typicality technique, in Chapter 8 of the thesis we also show that there are residually many Nikodym sets, i.e., sets of Lebesgue measure zero which contain a punctured hyperplane through every point of $\mathbb{R}^n$.

The Ph.D. thesis is based on the following papers

[H1] A. Chang, M. Csörnyei, K. Héra and T. Keleti, Small unions of affine subspaces and skeletons via Baire category, *Adv. Math.* **328** (2018), 801–821.

[H2] K. Héra, Hausdorff dimension of Furstenberg-type sets associated to families of affine subspaces, submitted to *Ann. Acad. Sci. Fenn. Math.*, arXiv:1809.04666.

[H3] K. Héra, T. Keleti and A. Máthé, Hausdorff dimension of unions of affine subspaces and of Furstenberg-type sets, to appear in *J. Fractal Geom.*, arXiv:1701.02299.

[H4] K. Héra, T. Keleti and A. Máthé, A Fubini-type theorem for Hausdorff dimension, in preparation

Additional scientific papers

[H5] M. Csörnyei, K. Héra and M. Laczkovich, Closed sets with the Kakeya property, *Mathematika* **63** (2017), 184–195.

[H6] K. Héra, M. Laczkovich, The Kakeya problem for circular arcs, *Acta Mathematica Hungarica* **150** (2016), 479–511.

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